UDC 532.527

THE MOST SIMPLE TYPE OF FLOWS WITH FORMATION OF TANGENTIAL DISCONTINUITIES

PMM Vol. 40, № 5, 1976, pp. 936, 941

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(Received August 12, 1974)

A method of solving certain problems of unsteady plane flows of an ideal fluid past a body with separation in terms of analytic series, is proposed. The form of expansion of the functions sought is obtained such, that the equations can be separated and the coefficients of expansion computed successively. An inverse problem is formulated and exact solutions constructed. In the real flows with separation the viscosity plays an important part and, as was pointed out in [1], no unanimous opinion exists as to whether separation can occur in an ideal fluid. The exact solutions corroborate that such a possibility exists.

The work represents an attempt at further development of analytic methods of investigation [2 — 5].

1. Elements of the general theory. In an ideal fluid the separation is accompanied by formation of a surface of discontinuity of the tangential velocity components. The circulation Γ of the velocity along the contour intersecting this surface only at a single point W, is generally different from zero, and in the plane flow with separation the surface of tangential discontinuity can be represented by the function $W(t, \Gamma)$, $0 \leqslant \Gamma \leqslant \Gamma_0$. Here W represents a point on a complex plane, t is time and Γ_0 is the circulation along a contour enclosing the whole discontinuity. Under such parametric formulation, the velocity of motion of the points of the tangential discontinuity is equal to half of the sum of the velocities of motion of the fluid particles on both sides of the discontinuity.

By U we denote the complex velocity of a flow past a body without separation. The surface of tangential discontinuity induces, in the presence of the body in question, additional complex velocity V, which is discontinuous, and at the points of discontinuity we define the mean velocity $\langle V \rangle$.

We formulate the problem and solve it using auxiliary complex planes. Let $z(W)$ for example, be a conformal mapping which transfers the outside of the body onto the right half-plane of z. Then we have

$$\langle V \rangle = \langle v \rangle \frac{dz}{dW}, \quad \langle v \rangle = \frac{1}{2\pi i} \int_{-\Gamma_0}^{+\Gamma_0} \frac{d\Gamma_*}{z - z_*}$$

Here the integration is carried out on the z-plane along the mapping of the surface of discontinuity and its reflection in the ordinate axis, Γ_* denotes the variable of integration and $z_* = z(t, \Gamma_*)$.

If separation takes place only at the sharp edges of the body, then the flows past the body with and without separation are in one-to-one correspondence, and the problem is reduced to that of determining the bounded complex velocity $U + V$ of the separation flow past in the specified complex velocity U without separation. To do this, we must solve the following system with respect to the unknowns $W(t, \Gamma)$ and $\Gamma_0(t)$:

$$(U + \langle V\rangle)^* = \frac{dW}{dt}, \quad |U + V| < \infty$$

(the upper asterisk denotes a complex conjugate). The above inequality represents a generalization of the Joukowski postulate and can be reduced to equations of various types. If e.g. the sharp edge of the boundary terminates in a segment of the straight line $(-\varepsilon, 0)$, the postulate can be written in the form

$$\mathrm{Im}\,(U + V) = 0, \quad \Gamma = \Gamma_0$$

The parameter Γ can conveniently be replaced by the time τ elapsed from the instant of appearance of the given point of the tangential discontinuity. In this case Γ becomes an unknown quantity depending on the difference $t - \tau$. The expression for $\langle v\rangle$ now becomes

$$\langle v\rangle = \frac{1}{2\pi i}\int_{-t}^{+t} \frac{d\tau_*}{z - z_*}\frac{\partial \Gamma_*}{\partial \tau_*}, \quad \Gamma_* = \Gamma(t - |\tau_*|)$$

2. Case of a semi-infinite plate. Let us consider a flow past a semi-infinite plate $(-\infty, 0)$ with separation, corresponding to a flow without separation, the complex velocity of which is

$$U = 1 + A(t)/(2iW^{1/2}) \tag{2.1}$$

Here $A(t)$ is an analytic function satisfying the condition $A(0) = 0$. In this case the function $z = \sqrt{W}$ maps the outside of the half-plate conformally onto the right half-plane of z and the problem is reduced to solving the system

$$\left(1 + \frac{A(t)}{2iz} + \frac{\langle v\rangle}{2z}\right)^* = \frac{dz^2}{dt} \tag{2.2}$$

$$\frac{A(t)}{i} + \langle v(t, 0)\rangle = 0, \quad v(t, \tau) = \frac{1}{2\pi i}\int \frac{\partial \Gamma}{\partial \tau_*}\frac{d\tau_*}{z - z_*} \tag{2.3}$$

The integration here is also carried out over the discontinuity and its reflection (indicated in Fig. 1 by a dashed line) $z_* = z(t, \tau_*)$. The problem of constructing the field of

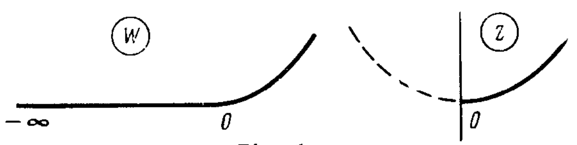

Fig. 1

flow is reduced to that of solving the system (2.2), (2.3) for the unknowns $z(t, \tau)$ and $\Gamma_0(t)$.

Eliminating $A(t)$ from (2.1) and changing the variables according to the formulas $t = t_1^2$, $\tau = \tau_1^2$ and $z = \tau_1(1 + \eta)$, we can reduce the system (2.2), (2.3) to the form

$$\eta^* = \left[1 + \tau_1^{-2}\int_\Gamma (\langle v(t_1, \tau_1)\rangle - \langle v(t_1, 0)\rangle)\frac{d\tau_1}{1 + \eta}\right]^{1/2} - 1 \tag{2.4}$$

$$\frac{A(t)}{i} + \langle v(t_1, 0)\rangle = 0, \quad v(t_1, \tau_1) = \frac{1}{2\pi i}\int_{-t_1}^{+t_1} \frac{\partial \Gamma}{\partial \tau_{1*}}\frac{d\tau_{1*}}{\tau_1(1 + \eta) - \tau_{1*}(1 + \eta_*)} \tag{2.5}$$

The integration is carried out on the t_1, τ_1 variable plane along the line $\Gamma = \mathrm{const}$, over the variable τ_1 within the interval $(0, \tau_1)$.

The method of solution can be conveniently explained using a semi-inverse problem in which the total circulation of the separated vortices is given in the form of an analytic function $M(t)$ and the quantity $A(t)$ is assumed to be unknown.

$$\Gamma_0(t) = t^{3/2} M(t) \tag{2.6}$$

The solution can be obtained by the iterative process in which the $(n+1)$-th iteration η_{n+1}^* is given by the right-hand side of (2.4) in which v is replaced by v_n and η by η_n ($\eta_0 = 0$); v_n is given by the right-hand side of the second equation of (2.5), where Γ is replaced by $\Gamma_0(t_1^2 - \tau_{1*}^2)$, η by η_n and η_* by η_{n*}. The iterative process has the following property: if the quantity $\eta_n = M_1(t_1^2, \tau_1)$ can also be written in the form of a convergent series in increasing powers of t_1^2 and τ_1, then $\eta_{n+1} = M_2(t_1^2, \tau_1)$ can also be written in the form of a convergent series in powers of t_1^2 and τ_1. From this it follows that the terms of the series M_2 are functions of the terms of the series M_1. But the terms of M_2 of order j, i.e. the terms of the form $a t_1^{2k} \tau_1^{j-2k}$ depend only on those terms of M_1 the order of which is strictly less than j. We therefore conclude that since the first order term of M_2 is independent of the terms of M_1, it will remain unchanged in all iterations beginning from the first iteration. Repeating this reasoning we see that the second order terms will also remain unchanged beginning from the second iteration since they depend on the first order term only, etc. The n-th order terms will remain unchanged beginning from the n-th iteration. It is clear that only these terms are important.

For example, for $\Gamma_0 = {}^1/_3 t^{3/2}$ we obtain

$$\eta = \frac{i}{12}\tau_1 - \frac{5}{288}\tau_1^2 + \frac{1}{96}t_1^2 + i\frac{29}{8640}\tau_1^3 - i\frac{7}{2304}t_1^2\tau_1 + \ldots$$

Changing to the x, y coordinates we can show that the discontinuity on the W-plane has the form

$$y(x, t) = x^{3/2} M_3(x, t) = \frac{1}{6}x^{3/2} - \frac{11}{1152}tx^{5/2} + \frac{41}{2880}x^{5/2} + \ldots$$

The coordinate x of the discontinuity moves according to the law

$$x = t - \frac{1}{48}t^2 + \ldots$$

The form of discontinuity is analytic near its end, but the distribution of the vorticity has the form $\Gamma = \rho^{3/2} M_4(t, \rho)$ where ρ is the distance of the particular point of the discontinuity from its end, and M_4 is an analytic function. We can also determine the function $A(t)$. In the case in question we have

$$A(t) = \frac{1}{4}t - \frac{1}{512}t^2 + \ldots$$

In solving the direct problem, we also bring Eq. (2.4) into the iterative process. It defines every time the total circulation Γ_{0n} of the separated vortices. The circulation can be written in the form (2.6) provided that the conditions imposed on the function $A(t)$ in (2.1) hold. The iteration which follows is the same as that in the semi-inverse problem. If, for example, $A(t) = t$, then

$$\eta = \frac{i}{3}\tau_1 - \frac{5}{18}\tau_1^2 + \frac{1}{6}t_1^2 + i\frac{4}{27}\tau_1^3 - i\frac{1}{9}t_1^2\tau_1 + \ldots$$

$$\Gamma_0 = \frac{4}{3}t^{3/2} + \frac{2}{15}t^{5/2} + \ldots$$

We note that the iterative process given above is based on the fact that the linearized part is used as the principal term. The process can also be used to determine a flow about a profile with separation, provided that the profile terminates in a straight line segment so that the separation-free flow can be expanded near the edge into a series with the principal terms (2. 1). In this case the outside of the profile maps onto the right half-plane of z, so that the point $z = 0$ corresponds to the edge and the problem reduces to that already discussed.

3. Formulation of the inverse problems. Proof of the theorem of existence of the discontinuity formation process can be carried out by proving the convergence of the series $\eta = M_5(t_1^2, \tau_1)$. This problem is however very complicated. We therefore present an indirect method utilizing the inverse problems, which are formulated as follows. We specify a process of formation of a discontinuity $W(t, \alpha)$, $\Gamma(t, \alpha)$, where α is a parameter, about the specified body. We require to find a vortex-free flow past the body, at which

$$(U + \langle V \rangle)^* = dW/dt \tag{3.1}$$

Next we must determine the sources of the given motion which can be generated by, for example, moving bodies.

To solve the inverse problems, we must first find dW/dt and $\langle V \rangle$, then use Eq. (3. 1) to find the function at the line of discontinuity and continue, if possible, this function analytically onto the complex plane. The inverse problem, just as the straight problem, is best solved using an auxiliary plane. For the given conformal mapping $W(z)$, the equation of motion assumes the following form on the auxiliary plane:

$$(u + \langle v \rangle)^* = |W|^2 \frac{dz}{dt}, \quad u = UWz$$

$$v = VWz, \quad \langle v \rangle = \langle V \rangle Wz$$

and the solution is obtained as before.

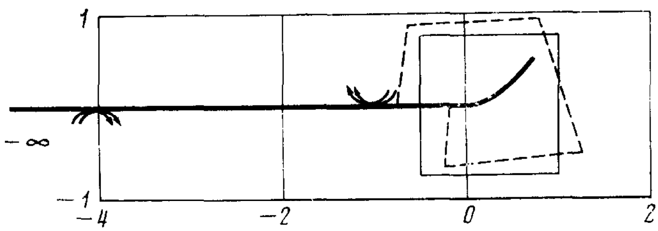

Fig. 2

Thus a solution of the inverse problem exists if the analytic continuation described above exists. Examining the problem 2 we can prove that the above analytic continuation always exists provided that the process of formation of the discontinuity is given in the form of the following convergent series:

$$\eta = M_5(t_1^2, \tau_1), \quad \Gamma_0 = t_1^{3/2} M(t_1)$$

Without indulging in a detailed discussion, we give two exact solutions obtained with the help of the above statement.

4. Exact solution of the problem with a discontinuity formed about a semi-infinite plate. We define the process of formation of a discontinuity about a semi-infinite plate $(-\infty, 0)$ (see Fig. 2) as follows:

$$W(\alpha) = 4 \sin^2 \alpha e^{i2\alpha}, \quad \Gamma(\alpha, t) = {}^2/_3 (t - 4 \operatorname{tg}^2 \alpha)^{3/2}, \qquad 0 \leqslant \alpha \leqslant \operatorname{arctg} ({}^1/_2 t^{1/2})$$

We define the auxiliary z -plane using the following pair of transformations:

$$\sqrt{W} = z_1, \quad \frac{2z_1}{2 + iz_1} = z$$

In the z -plane the discontinuity lies on the coordinate x. The resulting conformal transformation has the form

$$W(z) = \left(\frac{2z}{2 - iz}\right)^2$$

Writing the equation of motion of the discontinuity on the z-plane, we find

$$u = |\, Wz \,|^2\, dz/dt - \langle v \rangle \tag{4.1}$$

When $z = x$, we have

$$|\, Wz \,|^2 = 4x^2 (1 + {}^1/_4 x^4)^{-3}, \quad dz/dt = {}^1/_2 x^{-1}$$

$$\langle v \rangle = -i\,(x^2 - {}^1/_2 t) \quad (v = i\,(z^2 - t)^{1/2} z - i\,(z^2 - {}^1/_2 t))$$

On substituting the above expressions into (4.1) with $z = x$, we find $u = u\,(x, t)$. Obviously, for the analytic continuation it is sufficient to replace x by z. As the result we obtain

$$u = 2z\,(1 + {}^1/_4 z^4)^{-3} + i\,(z^2 - {}^1/_2 t)$$

The function $U + V = (u + v)/Wz$ which yields the solution of the problem, has a singularity on the upper surface of the plate at the point -1, and at the lower surface at the point -4 (see Fig. 2 in which arrows indicate the trajectories of the fluid particles). The solution is bounded within the square shown in Fig. 2.

We find the motion source in the following manner. We follow the movement of the fluid particles lying on the square contour (see Fig. 2) at the instant $t = 0$, to find the law of deformation of this contour. We can now assume that no flow takes place outside this contour. The contour itself filled with an ideal fluid is deformed under the action of the external forces in accordance with the law determined previously.

5. Exact solution of the problem with a discontinuity formed about a plate of finite width. We define the process of formation of a discontinuity about a plate $(-2, 0)$ (see Fig. 3) as follows:

$$W(\alpha) = 2 \sin^2 \alpha e^{i2\alpha} / (1 - \sin^2 \alpha e^{i2\alpha})$$

$$\Gamma(\alpha, t) = {}^2/_3 (t - 2\,\mathrm{tg}^2\, \alpha)^{3/2}, \quad 0 \leqslant \alpha \leqslant \mathrm{arctg}\, (t/2)^{1/2}$$

The following sequence of mappings defines the auxiliary z -plane and the resulting conformal mapping:

$$\frac{2W}{2 + W} = z_1, \quad \sqrt{z_1} = z_2, \quad \frac{\sqrt{2}\, z_2}{\sqrt{2} + iz_2} = z$$

$$W(z) = z^2 / (1 - i\sqrt{2} z - z^2)$$

As in the previous example, the discontinuity lies at the coordinate x and we have

$$|\, Wz \,|^2 = (4x^2 + 2x^4)(x^4 + 1)^{-2}, \quad dz\,/\,dt = {}^1/_2 x^{-1}$$

$$u(z, t) = (2z + z^3)(z^4 + 1)^{-2} + i\,(z^2 - {}^1/_2 t)$$

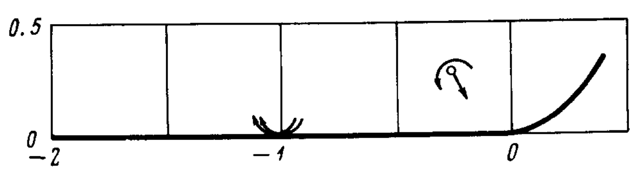

Fig. 3

The function $U + V$ providing the solution has a singularity at the point $-1/4 + 1/4i$, and at this point we have a vortex of intensity $\pi/2$ and a dipole. Arrows in Fig. 3 indicate the direction of the dipole axis and of the vortex circulation. Another singularity exists at the upper surface of the plate at the point -1 in the form of a dipole of variable intensity, and a fourth order pole of constant intensity. The velocity at infinity is directed at a certain angle to the plate. The circulation about the system plate — discontinuity. is absent. A usual singularity corresponding to the overflow of the fluid is present at the tip of the plate (see Fig. 3).

At the initial instant of time the velocity at infinity, the pole at the plate, the vortex and the dipole, induce a flow which has no singularities at the trailing edge. On the upper surface of the plate the dipole tends to cause the overflow and singularity at the trailing edge. These deviations from the regular character are neutralized by the tangential discontinuity which is formed.

The exact solution in the z-plane can be written in the form

$$u + v = (2z + z^3)(z^4 + 1)^{-2} + i\,(z^2 - t)^{1/2}z$$

On the W-plane the solution is much more complicated.

In both the above cases the discontinuity was assumed to be stationary, with only its length subject to change. This enabled us to obtain exact solutions in a very simple manner.

The author thanks A. A. Nikol'skii for formulating the basic problems and for the assistance given.

REFERENCES

1. Sedov, L. I., Plane Problems of the Hydrodynamics and Aerodynamics. Moscow, "Nauka", 1966.
2. Nikol'skii, A. A., On the "second" form of motion of an ideal fluid about a body (investigation of separated vortex flows). Dokl. Akad. Nauk SSSR, Vol. 116, № 2, 1957.
3. Nikol'skii, A. A., Action of the "second" form of hydrodynamic motion on flat bodies (dynamics of separated plane flows). Dokl. Akad. Nauk SSSR, Vol. 116, № 3, 1957.
4. Nikol'skii, A. A., Similarity law for three-dimensional stationary separated flows of liquids and gases past bodies. Uch. zap. TsAGI, Vol. 1, 1970.
5. Nikol'skii, A. A., Nonlinear similarity law for separated supersonic flow of an ideal gas past a rectangular wing. Uch. zap. TsAGI, Vol. 3, № 6, 1972.

Translated by L. K.